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The double torus as a 2D cosmos: groups, geometry and closed geodesics

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Abstract

The double torus provides a relativistic model for a closed 2D cosmos with topology of genus 2 and constant negative curvature. Its unfolding into an octagon extends to an octagonal tessellation of its universal covering, the hyperbolic space H^2. The tessellation is analysed with tools from hyperbolic crystallography. Actions on H^2 of groups/subgroups are identified for $SU(1, 1)$, for a hyperbolic Coxeter group acting also on $SU(1, 1)$, and for the homotopy group Φ_2 whose extension is normal in the Coxeter group. Closed geodesics arise from links on H^2 between octagon centres. The direction and length of the shortest closed geodesics is computed.

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Dedicated to Marcos Moshinsky on the occasion of his 80th birthday.

1. Introduction

In a publication [7] entitled Cosmic Topology, Lachieze-Rey and Luminet review topological alternatives for pseudo-Riemannian manifolds of constant negative curvature as cosmological models and discuss their implications for observations. The general classification of spaces of constant curvature was given by Wolf [11]. Some of these manifolds admit closed (null-) geodesics. Test particles (photons) travelling along such geodesics produce images of their own source and therefore observable effects under conditions described in [7, pp 189–202]. All these simple geometric models of a cosmos neglect the influence of varying mass distributions. These distort locally the curvature and the geodesics assumed in the models for test particles, see [7, p 173]. Of particular interest for observing the topology, see [7] for details, are the shortest closed geodesics in at least three ways:

(i) They produce the brightest images of a source and are considered less sensitive to the local distortions.
(ii) They determine the first peaks in the autocorrelation function of the matter distribution.

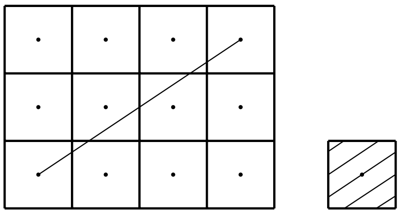

Figure 1. The universal covering manifold of the torus is a plane E^2 tessellated by unit squares. The homotopy group $Z \times Z$ of the torus acts on E^2 by translations (m, n). The Euclidean metric on E^2 induces on the torus a Riemannian metric. Geodesic sections between centres of squares on E^2 become closed geodesics on the torus. On the left, a geodesic line on E^2 of slope $2/3$ and of length $s = \sqrt{3^2 + 2^2}$ connects the centres $(0, 0)$ and $(3, 2)$ on E^2. On the right, this geodesic is pulled back to the initial square.

(iii) By their length they determine a global radius of the model. If no correspondence between sources and images can be observed, this puts a lower limit on this global radius.

For comparison we sketch an analysis for the 2D torus manifold with Riemannian metric. Its fundamental polygon can be taken as a unit square. Its universal covering manifold is a plane which admits a tessellation by copies of one fundamental initial square. The Euclidean metric on the plane E^2 induces on the torus a Riemannian metric with zero curvature. The homotopy group of the torus is $Z \times Z$. This homotopy group is isomorphic to a 2D translation group acting on the plane whose elements (m, n) yield the square lattice of centre positions of all squares, seen from the initial square. The straight lines on the plane, when pulled back to the reference square, determine its geodesics. Consider now a straight line passing through the centre of the initial square. If and only if it hits for the first time the centre (m, n) of another square, then m, n are relatively prime, and the geodesic when pulled back to the initial square will close on the torus. The slope of this closed geodesic is n/m, its length is $s = \sqrt{m^2 + n^2}$. The problem of closed geodesics on the torus is thus solved by simple crystallographic computations on the plane. These relations are illustrated in figure 1. The shortest closed geodesics of length $s = 1$ clearly run between the centres of the initial square and its four edge neighbours. By giving the squares a general fixed edge length one could introduce a global radius of this torus model.

In what follows we examine the continuous and discrete groups and the geometry for the 2D double torus as a paradigm for a cosmos with closed geodesics. The analysis will in concept follow similar steps as sketched above for the torus. In contrast and in detail it will involve the action of crystallographic groups on hyperbolic space with pseudo-Riemannian metric, a field in which the present authors have been interested [5,6]. We believe that similar techniques apply to some other topologies of [7].

We start from the information on the double torus given in [7]. The fundamental domain of the double torus is a hyperbolic octagon. It was described by Hilbert and Cohn-Vossen in 1932 [2]. The universal covering manifold of the double torus must admit a tessellation by these octagons. This topological condition enforces the hyperbolic space H^2 as the universal covering of the double torus. The homotopy group of the double torus as discussed by Seifert and Threlfall [10, pp 6, 174] and by Coxeter [1, p 59] can be converted into a group Φ_2 acting on the universal covering manifold H^2. The generators and the single relation for this group were given by Coxeter [1] and Magnus [8]. On the universal covering manifold we shall see that the search for pre-images of closed geodesics becomes a problem of hyperbolic crystallography.

In section 2 we describe the group $SU(1, 1)$, its action and relevant cosets. In section 3 we review the universal covering of the double torus, the hyperbolic space $H^2 \sim SU(1, 1)/U(1)$.

Equipped with a pseudo-Riemannian structure it has constant negative curvature and is invariant under the action of $SU(1,1)$. In section 4 we give the general group description and group action for H^2. The octagonal fundamental domain of the double torus is identified as a double coset of $SU(1,1)$ with respect to the left action of the homotopy group Φ_2 and the right action of $U(1)$. In section 5 we analyse the homotopy group as a subgroup of a hyperbolic Coxeter group. An extension of the homotopy group is shown to be a normal subgroup of this hyperbolic Coxeter group which by conjugation yields the octagonal symmetry. In section 6 we turn to the geometric action of the homotopy group. A geometric origin of the relation between the generators is given. We characterize on $SU(1,1)/U(1)$ the condition for closed geodesics. We determine the next-neighbour octagons in the tessellation and from them find the shortest closed geodesics on the double torus.

2. Groups and actions on Minkowski space $M(1,2)$

Here we describe the group $SU(1,1)$, its universal covering relation to $SO_{\uparrow}(1,2)$, and their action on Minkowski space $M(1,2)$.

2.1. The group $SU(1,1)$

We first discuss group elements g of $SU(1,1)$ and their parameters:

$$\begin{aligned}
&g \in SU(1,1) : gMg^{\dagger}M = e \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad M^2 = e \\
&g = \begin{bmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{bmatrix} \qquad |\lambda|^2 - |\mu|^2 = 1 \\
&\lambda = \xi_0 + \mathrm{i}\xi_1 \qquad \mu = \xi_3 - \mathrm{i}\xi_2.
\end{aligned} \tag{1}$$

Here ξ_j are real parameters.

Exponential parameters arise as follows: Define in terms of the Pauli matrices the 2×2 matrix representation of the Lie algebra $su(1,1)$,

$$\begin{aligned}
&\tau_0 := \mathrm{i}\sigma_3 \qquad \tau_1 := \sigma_2 \qquad \tau_2 := \sigma_1 \\
&-\tau_0^2 = \tau_1^2 = \tau_2^2 = 1 \\
&g = \exp(\tilde{\alpha}h) \qquad h + Mh^{\dagger}M = 0 \qquad \tilde{\alpha} \text{ real} \\
&h = \sum_0^2 \omega_j\tau_j = \begin{bmatrix} \mathrm{i}\omega_0 & \omega_2 - \mathrm{i}\omega_1 \\ \omega_2 + \mathrm{i}\omega_1 & -\mathrm{i}\omega_0 \end{bmatrix} \\
&(\tilde{\alpha}h)^2 = -(\tilde{\alpha})^2(\omega_0^2 - \omega_1^2 - \omega_2^2)e.
\end{aligned} \tag{2}$$

With the exponential parameters, elements of $SU(1,1)$ are described by an angle $\tilde{\alpha}$ and by the vector $\omega = (\omega_0, \omega_1, \omega_2)$ in $M(1,2)$, see equation (8). Depending on the value of $|\operatorname{tr}(g)|/2 = |(\lambda + \bar{\lambda})|/2 : \langle <1, >1, =1 \rangle$ we have three cases, the elliptic case

$$\begin{aligned}
&1 = (\omega_0^2 - \omega_1^2 - \omega_2^2) \\
&g = \cos(\tilde{\alpha})e + \sin(\tilde{\alpha})(\omega_0\tau_0 + \omega_1\tau_1 + \omega_2\tau_2) \\
&\xi_0 = \cos(\tilde{\alpha}) \qquad \xi_j = \sin(\tilde{\alpha})\omega_{j-1}
\end{aligned} \tag{3}$$

the hyperbolic case

$$\begin{aligned}
&-1 = (\omega_0^2 - \omega_1^2 - \omega_2^2) \\
&g = \cosh(\tilde{\alpha}) + \sinh(\tilde{\alpha})(\omega_0\tau_0 + \omega_1\tau_1 + \omega_2\tau_2) \\
&\xi_0 = \cosh(\tilde{\alpha}), \qquad \xi_j = \sinh(\tilde{\alpha})\omega_{j-1}
\end{aligned} \tag{4}$$

and the Jordan case

$$
\begin{aligned}
&0 = (\omega_0^2 - \omega_1^2 - \omega_2^2) \\
&g = e + (\omega_0\tau_0 + \omega_1\tau_1 + \omega_2\tau_2) \qquad (5)\\
&\xi_0 = 1 \qquad \xi_j = \omega_{j-1}
\end{aligned}
$$

where the real parameters ξ_j of equation (1) are identified in each case.

An element of elliptic type may be written, see equations (10) and (12), as

$$
\begin{aligned}
g &= r_3(\alpha)b_2(\theta)r_3(\gamma)r_3(\tilde{\alpha})r_3(-\gamma)b_2(-\theta)r_3(-\delta) \\
&= \begin{bmatrix} \cos(\tilde{\alpha}) + \mathrm{i}\cosh(2\theta)\sin(\tilde{\alpha}) & -\mathrm{i}\sinh(2\theta)\sin(\tilde{\alpha})\exp(2\mathrm{i}\alpha) \\ \mathrm{i}\sinh(2\theta)\sin(\tilde{\alpha})\exp(-2\mathrm{i}\alpha) & \cos(\tilde{\alpha}) - \mathrm{i}\cosh(2\theta)\sin(\tilde{\alpha}) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\tilde{\alpha}) + \mathrm{i}\sin(\tilde{\alpha})\omega_0 & \sin(\tilde{\alpha})(\omega_2 - \mathrm{i}\omega_1) \\ \sin(\tilde{\alpha})(\omega_2 + \mathrm{i}\omega_1) & \cos(\tilde{\alpha}) - \mathrm{i}\sin(\tilde{\alpha})\omega_1 \end{bmatrix} \qquad (6)
\end{aligned}
$$

$\omega = (\cosh(2\theta), \sinh(2\theta)\cos(2\alpha), \sinh(2\theta)\sin(2\alpha))$.

The adjoint action of $SU(1,1)$ on its Lie algebra can be written by use of equation (1) as

$$
\begin{aligned}
&(g, h) \to ghg^{-1} \\
&(g, -\mathrm{i}hM) \to g(-\mathrm{i}h)g^{-1}M = g(-\mathrm{i}hM)g^{\dagger} \qquad (7)\\
&(-\mathrm{i}hM)^{\dagger} = -\mathrm{i}hM.
\end{aligned}
$$

By the adjoint action, the vector components $\omega(g) = (\omega_0, \omega_1, \omega_2)$ associated with g are linearly transformed according to the homomorphism $SU(1,1) \to SO_{\uparrow}(1,2)$, see section 2.2. Any vector $\omega(g)$ is fixed under the action of g. The vectors ω for elliptic, hyperbolic and Jordan case correspond to and transform as time-like, space-like and light-like vectors on $M(1,2)$.

2.2. Group action and coset spaces

We adopt a vector and matrix notation in Minkowski space $M(1,2)$:

$$
\begin{aligned}
&x = (x_0, x_1, x_2) \\
&\langle x, y\rangle := x_0y_0 - x_1y_1 - x_2y_2 \qquad (8)\\
&\tilde{x} := \begin{bmatrix} x_0 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 \end{bmatrix}.
\end{aligned}
$$

The group action on $M(1,2)$ for $g \in SU(1,1)$ we define by

$$
\begin{aligned}
&(g, \tilde{x}) \to g\tilde{x}g^{\dagger} \\
&(g, x_i) \to (gx)_i = \sum_j L_{ij}(g)x_j. \qquad (9)
\end{aligned}
$$

This action preserves hermiticity, the determinant, and hence the metric on $M(1,2)$. $L(g)$ with $L(-g) = L(g)$ is the two-to-one homomorphism of $SU(1,1)$ to $SO_{\uparrow}(1,2)$, the orthochronous Lorentz group, see [4]. We have chosen the action of $SU(1,1)$ on $M(1,2)$ in equation (9) to agree with the adjoint action equation (7) on the Lie algebra with the exponential parameters equation (2). We define the elements

$$
\begin{aligned}
&r_3(\alpha) = \begin{bmatrix} \exp(\mathrm{i}\alpha) & 0 \\ 0 & \exp(-\mathrm{i}\alpha) \end{bmatrix} \qquad \omega = (1, 0, 0) \\
&b_2(\theta) = \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix} \qquad \omega = (0, 0, 1) \qquad (10)
\end{aligned}
$$

of $SU(1,1)$ whose homomorphic images

$$
\begin{aligned}
L_3(2\alpha) = L(r_3(\alpha)) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2\alpha) & -\sin(2\alpha) \\ 0 & \sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \\
L_2(2\theta) = L(b_2(\theta)) &= \begin{bmatrix} \cosh(2\theta) & \sinh(2\theta) & 0 \\ \sinh(2\theta) & \cosh(2\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{11}
$$

are a rotation from $U(1)$ or a boost respectively. The factor 2 in the angular parameters reflects our emphasis on the group $SU(1,1)$. A general group element of $SU(1,1)$ admits the Euler-type parametrization

$$g = g(\Omega) = g(\alpha, \theta, \gamma) := r_3(\alpha) b_2(\theta) r_3(\gamma). \tag{12}$$

We shall need the multiplication rule for two elements g_1, g_2 written in these parameters. A special case of this multiplication is

$$
\begin{aligned}
& g_1(0, \theta_1, \gamma_1) g_2(0, \theta_2, 0) = g(\alpha, \theta, \gamma) \\
& \cosh(2\theta) = \cosh(2\theta_1)\cosh(2\theta_2) + \sinh(2\theta_1)\sinh(2\theta_2)\cos(2\gamma_1) \\
& \sinh(2\theta)\sin(2\alpha) = \sinh(2\theta_2)\sin(2\gamma_1) \\
& \sinh(2\theta)\sin(2\gamma) = \sinh(2\theta_1)\sin(2\gamma_1).
\end{aligned}
\tag{13}
$$

For general products $g_1(\alpha_1, \theta_1, \gamma_1) g_2(\alpha_2, \theta_2, \gamma_2)$ we conclude from equation (13) that

$$\cosh(2\theta(g_1 g_2)) = \cosh(2\theta_1)\cosh(2\theta_2) + \sinh(2\theta_1)\sinh(2\theta_2)\cos(2(\gamma_1 + \alpha_2)). \tag{14}$$

The general multiplication law is obtained from equation (14) by right- and left-multiplications with rotation matrices of type $r_3(\alpha)$.

The hyperbolic space H^2, see section 3, is the coset space $SU(1,1)/U(1)$. It is a homogeneous space under the action of $SU(1,1)$. The points of $SU(1,1)/U(1)$ we choose as the elements $c \in SU(1,1)$ whose Euler angle parameters from equation (12) obey $\gamma(c) = 0$. Any element $g \in SU(1,1)$ can now be written as

$$g = ch \qquad h \in U(1). \tag{15}$$

The vectors $x \in M(1,2)$, $\langle x, x \rangle = 1$ of H^2, and the coset elements c are in one-to-one correspondence, $x \leftrightarrow c$.

2.3. Involutive automorphisms of $SU(1,1)$

There are two involutive automorphisms ψ_1, ψ_2, $\psi_1^2 = e, \psi_2^2 = e$ of $SU(1,1)$:

$$
\begin{aligned}
& (\psi_1, g) \to \bar{g} \\
& (\psi_2, g) \to (g^\dagger)^{-1}.
\end{aligned}
\tag{16}
$$

The action of these automorphisms is given explicitly by

$$
\begin{aligned}
\psi_1\left(\begin{bmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{bmatrix}\right) &= \begin{bmatrix} \bar{\lambda} & \bar{\mu} \\ \mu & \lambda \end{bmatrix} \\
\psi_2\left(\begin{bmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{bmatrix}\right) &= \begin{bmatrix} \lambda & -\mu \\ -\bar{\mu} & \bar{\lambda} \end{bmatrix}.
\end{aligned}
\tag{17}
$$

The stability subgroups H_{ψ_i} under these automorphisms are easily seen to be $H_{\psi_1} = SO(1,1) = \langle b_2(\theta) \rangle$, $H_{\psi_2} = U(1) = \langle r_3(\alpha) \rangle$. It follows that the automorphisms must map the coset spaces with respect to these subgroups into themselves.

We shall combine these automorphisms together with the left action $(g_1, g) \to g_1 g$ into a larger group. This larger group admits the involutions $\psi_1(g_1) := g_1 \circ \psi_1 \circ g_1^{-1}$, $\psi_2(g_2) := g_2 \circ \psi_2 \circ g_2^{-1}$ which act on $SU(1,1)$ as

$$\begin{aligned}(\psi_1(g_1), g) &\to (g_1 \circ \psi_1 \circ g_1^{-1})g = g_1(\bar{g}_1)^{-1}\bar{g} \\ (\psi_2(g_2), g) &\to (g_2 \circ \psi_2 \circ g_2^{-1})g = g_2 g_2^{\dagger}(g^{\dagger})^{-1}.\end{aligned} \tag{18}$$

Here and in what follows we use the symbol $\circ$ to emphasize operator multiplication. From the stability groups for ψ_1, ψ_2 it follows that all $\psi_1(g_1)$, $\psi_2(g_2)$ are in one-to-one correspondence to the points of the cosets $g_1 \in SU(1,1)/SO(1,1)$, $g_2 \in SU(1,1)/U(1)$.

2.4. Weyl reflections and involutive automorphisms

A Weyl reflection acting on $M(1,2)$ with a Weyl vector k is defined as the involutive map

$$(W_k, x) \to y = x - 2(\langle x, k\rangle / \langle k, k\rangle)\, k. \tag{19}$$

It preserves the scalar product equation (8), and leaves any point x of the plane $\langle k, x\rangle = 0$ fixed. Weyl reflections act on the coset spaces in $M(1,2)$. We can distinguish two types of Weyl reflections for k time-like or space-like. All Weyl operators W_k in $M(1,2)$ with k space-like may be expressed with

$$\begin{aligned}k &= L(r_3(\alpha))L(b_2(\theta))(0,1,0) \\ &= L(r_3(\alpha))L(b_2(\theta))L(r_3(\pi/4))(0,0,-1) \\ &= (\sinh(2\theta), \cosh(2\theta)\cos(2\alpha), \cosh(2\theta)\sin(2\alpha)).\end{aligned} \tag{20}$$

Here we let $L(g)$ represent actions $g \in SU(1,1)$ on $M(1,2)$. As representative it proves convenient to pass to $k^0 = (0,0,-1)$ with

$$(W_{(0,0,-1)}, (x_0, x_1, x_2)) \to (x_0, x_1, -x_2). \tag{21}$$

For Weyl reflections we have the general conjugation property

$$L(g) \circ W_k \circ L(g^{-1}) = W_{L(g)k} \tag{22}$$

which applies in particular to $k^0 = (0,0,-1)$.

We now wish to define operators acting on $SU(1,1)$ by left-multiplication and by involutive automorphisms which correspond to Weyl reflections. In the disc model of section 3.2, the Weyl reflection of equation (21) corresponds to complex conjugation, and we therefore shall associate with $k^0 = (0,0,-1)$ the automorphism ψ_1 of equation (18). For operator products we have the identity

$$\psi_1 \circ g \circ \psi_1 = \bar{g}. \tag{23}$$

It allows to convert operator products containing an even number of automorphisms into pure left-multiplications. Guided by equations (20), (22) we define an involutive automorphism corresponding to a general Weyl vector k by forming in correspondence to equation (22) the operator product

$$\begin{aligned}s_k &:= g \circ \psi_1 \circ g^{-1} = g \circ \psi_1 \circ g^{-1} \circ \psi_1 \circ \psi_1 = g(\bar{g})^{-1} \circ \psi_1 \\ g &= r_3(\alpha)b_2(\theta)r_3(\pi/4) \\ g(\bar{g})^{-1} &= r_3(\alpha)b_2(\theta)r_3(\pi/2)b_2(-\theta)r_3(\alpha) \\ &= \begin{bmatrix} \mathrm{i}\cosh(2\theta)\exp(2\mathrm{i}\alpha) & -\mathrm{i}\sinh(2\theta) \\ \mathrm{i}\sinh(2\theta) & -\mathrm{i}\cosh(2\theta)\exp(-2\mathrm{i}\alpha) \end{bmatrix}.\end{aligned} \tag{24}$$

The matrix g is taken from equation (20) but expressed on the level of $SU(1,1)$.

3. The two-dimensional hyperbolic manifold H^2

In this section we introduce the hyperbolic manifold H^2 as a hyperboloid in $M(1, 2)$, the hyperbolic disc model, and review the pseudo-Riemannian structure on H^2.

3.1. H^2 as a hyperboloid

In $M(1, 2)$ we consider the hyperboloid $H^2 : \langle x, x\rangle = x_0^2 - x_1^2 - x_2^2 = 1,\ x_0 > 0$, see Ratcliffe [9, pp 56–104]. Its representative point $x^0 = (1, 0, 0)$ has the stability group $U(1) \sim SO(2, R)$ generated by rotations. Therefore, the hyperbolic manifold H^2 is the coset space $SU(1, 1)/U(1)$. A parametrization of H^2 is obtained from equations (8), (9):

$$\begin{aligned} x &= L(r_3(\alpha)b_2(\theta))x^0 = (\cosh(2\theta), \sinh(2\theta)\cos(2\alpha), \sinh(2\theta)\sin(2\alpha)) \\ \tilde{x} \to \tilde{x}(\theta, \alpha) &= r_3(\alpha)b_2(\theta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} b_2^\dagger(\theta) r_3^\dagger(\alpha) \\ &= \begin{bmatrix} \cosh(2\theta) & \sinh(2\theta)\exp(2\mathrm{i}\alpha) \\ \sinh(2\theta)\exp(-2\mathrm{i}\alpha) & \cosh(2\theta) \end{bmatrix}. \end{aligned} \tag{25}$$

The action of $SU(1, 1)$ on H^2 is given in equations (8), (9). We introduce in $M(1, 2)$ intersections of the coset space $SU(1, 1)/U(1)$ with planes through $(0, 0, 0)$ characterized by their normal vector k, $\langle k, x\rangle = 0$. These intersections exist only if k is space-like. They always define geodesics, see [9, pp 68–70]. For $k = (0, 1, 0)$ we can write the intersection points as $(x_0, x_1, x_2) = (\cosh(2\rho), 0, \sinh(2\rho))$. Then 2ρ is a geodesic length parameter, see section 3.2. The application of $r_3(\alpha)b_2(\theta)$ to k and x from equations (10) and (11) yields

$$\begin{aligned} &k \to (\sinh(2\theta), \cosh(2\theta)\cos(2\alpha), \cosh(2\theta)\sin(2\alpha)) \\ &x(\rho) = (\cosh(2\theta)\cosh(2\rho), \cos(2\alpha)\sinh(2\theta)\cosh(2\rho) \\ &\qquad - \sin(2\alpha)\sinh(2\rho), \sin(2\alpha)\sinh(2\theta)\cosh(2\rho) + \cos(2\alpha)\sinh(2\rho)). \end{aligned} \tag{26}$$

All the vectors k normal to planes which intersect H^2 are space-like and may be written in terms of (α, β). Geodesics are mapped into geodesics under $SU(1, 1)$. We have the following.

Proposition 1. The expression equation (26) yields the most general geodesics on H^2 in a form parametrized by the geodesic length 2ρ. If we wish to determine the geodesic between two given time-like points $x^1 \neq x^2$, we can determine the normal and the parametrized geodesic from the vector product [9, pp 64–6] $k \sim (x^1 \times x^2)$, which is perpendicular to x^1, x^2, by an appropriate normalization and parametrization.

3.2. The hyperbolic disc

The hyperbolic disc parametrization of H^2, see [9, pp 127–35], is given by the map of the hyperboloid to the interior of the unit circle,

$$x, \langle x, x\rangle = 1 \to z = (x_1 + \mathrm{i}x_2)/(1 + x_0) \qquad x_0 \geqslant 1 \quad |z| \leqslant 1 \tag{27}$$

with inverse

$$x_0 = (1 + |z|^2)/(1 - |z|^2) \qquad x_1 + \mathrm{i}x_2 = z\, 2/(1 - |z|^2). \tag{28}$$

The map from the hyperboloid to the hyperbolic disc is conformal [9, p 8], i.e. preserves the angle between geodesics. The hyperbolic disc represents $SU(1, 1)/U(1)$ within $M(1, 2)$.

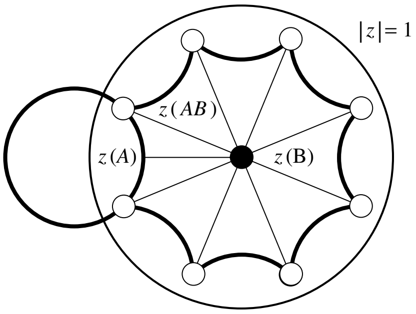

Figure 2. The octagon that forms the fundamental domain of the double torus in the unit circle $|z| = 1$ of the hyperbolic disc model. The eight edges are parts of geodesic circles as indicated on the left-hand side. The vertices of the fundamental hyperbolic Coxeter triangle are marked by the complex numbers $z(B)$, $z(AB)$, $z(A)$ which represent the rotation axes of these elliptic elements in the hyperbolic disc model.

The action of Lorentz transformations on the disc is nevertheless described by the group $SU(1, 1)$. It is given by the linear fractional transform

$$\begin{aligned} g &= \begin{bmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{bmatrix} \\ (g, z) &\to w = \frac{\lambda z + \mu}{\bar{\mu} z + \bar{\lambda}}. \end{aligned} \tag{29}$$

Extend the complex variable z to the full complex plane C and consider a circle of radius R centred at $q \in C$. Its equation is

$$(\overline{z - q})(z - q) - R^2 = \bar{z}z - \bar{q}z - q\bar{z} + \bar{q}q - R^2 = 0. \tag{30}$$

Define the vector $k = k_0(1, \mathrm{Re}\,(q), \mathrm{Im}\,(q))$, $k_0 > 1$. The vector k is space-like if $|q|^2 \geqslant 1$. For the intersection of the plane perpendicular to k with H^2 we find with equations (27), (28)

$$0 = \langle k, x\rangle = \tfrac{1}{2} k_0 (1 + x_0)(\bar{z}z - \bar{q}z - q\bar{z} + 1). \tag{31}$$

Comparing equation (30) we find: the geodesic intersection of the plane perpendicular to space-like k in the hyperbolic disc model is a circle with centre $q = q(k)$, $|q|^2 \geqslant 1$ and of radius $R = \sqrt{|q|^2 - 1}$. Comparison with the unit circle $|z|^2 = 1$ shows that the two circles have perpendicular intersections. An example of a geodesic circle is given in figure 2. This circle contains an edge line of an octagon.

Since planar intersections in the hyperboloid model map into circles in the hyperbolic disc model, we look for the action of Weyl reflections on the hyperbolic disc. They must be reflections in the circles.

Proposition 2. The Weyl reflection with space-like unit Weyl vector k, $\langle k, k\rangle = -1$ in the hyperbolic disc model determines a non-analytic fractional map

$$\begin{aligned} &q = (k_1 + \mathrm{i}k_2)/(k_0) = (\cosh(2\theta)/(\sinh(2\theta)) \exp(2\mathrm{i}\alpha) \\ &k = k_0(1, \mathrm{Re}\,(q), \mathrm{Im}\,(q)) \qquad k_0 = R^{-1} = 1\Big/\left(\sqrt{\bar{q}q - 1}\right) \\ &(W_k, z) \to w = s_k z = g\bar{g}^{-1}\psi_1 z = g\bar{g}^{-1}\bar{z} = \frac{a\bar{z} + b}{c\bar{z} + d} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = g\bar{g}^{-1} \end{aligned} \tag{32}$$

where $g\bar{g}^{-1}$ is given in equation (24).

Proof. The expression equation (32) arises by application of the automorphism equation (24): the automorphism ψ_1 acting on the hyperbolic disc yields $z \rightarrow \bar{z}$ and is followed by the fractional transform with the matrix given in equation (24).

In the hyperbolic disc model, any geodesic circle divides the unit circle into two parts. These two parts are mapped into one another under a Weyl reflection with space-like Weyl vector. Given two points $z^1, z^2,\ z^1 \neq 0$ in the hyperbolic disc model, the unique geodesic circle which passes through both of them can be obtained by inserting z^1, z^2 into equation (30) and solving for the centre q. One finds

$$q = q(z^1, z^2) = (\bar{z}^1 z^2 - z^1 \bar{z}^2)^{-1}(z^2 - z^1 - z^1 z^2 \overline{(z^2 - z^1)}). \qquad (33)$$

In the case $\bar{z}^1 z^2 - z^1 \bar{z}^2 = 0$ when equation (33) does not apply the geodesic is a straight line $z = \tanh(\rho)\ z^1/|z^1|$ through the origin of the unit disc.

3.3. Pseudo-Riemannian structure and curvature on H^2

For the notation we refer to [7, pp 146–7]. On 2D surfaces in $M(1,2)$ with coordinates u^μ we define a pseudo-Riemannian metric with line element

$$\begin{aligned} \mathrm{d}s^2 &= \sum_{\mu\nu} g_{\mu\nu}\, \mathrm{d}u^\mu\, \mathrm{d}u^\nu \\ &= \sum_{\mu,\nu} \left(\frac{\partial x_0}{\partial u^\mu}\frac{\partial x_0}{\partial u^\nu} - \frac{\partial x_1}{\partial u^\mu}\frac{\partial x_1}{\partial u^\nu} - \frac{\partial x_2}{\partial u^\mu}\frac{\partial x_2}{\partial u^\nu} \right) \mathrm{d}u^\mu\, \mathrm{d}u^\nu. \end{aligned} \qquad (34)$$

On H^2 with coordinates equation (25) we obtain with $u^1 = 2\theta, u^2 = 2\phi$

$$\begin{aligned} g_{..} &= (g_{\mu\nu}) = \begin{bmatrix} -1 & 0 \\ 0 & -(\sinh(2\theta))^2 \end{bmatrix} \\ g^{..} &= (g^{\mu\nu}) = \begin{bmatrix} -1 & 0 \\ 0 & -(\sinh(2\theta))^{-2} \end{bmatrix} \\ \mathrm{d}s^2 &= -(2\,\mathrm{d}\theta)^2 - (\sinh(2\theta))^2\, \mathrm{d}(2\phi)^2 \leqslant 0. \end{aligned} \qquad (35)$$

For any curve $\theta = \rho, \phi = c_0$ on H^2 with parameter ρ we find

$$\left(\frac{\mathrm{d}s}{\mathrm{d}(2\rho)}\right)^2 = -1 \qquad (36)$$

so that 2ρ is a geodesic length parameter. General geodesics with the same length parameter can then be constructed as given in equation (26).

In general relativity, the geodesics are the world lines followed by free test particles. Null geodesics $\mathrm{d}s^2 = 0$ are followed by photons, and geodesics $\mathrm{d}s^2 > 0$ are world lines for massive test particles. Note that all points on H^2 have space-like separation and so can be connected only by geodesics with $\mathrm{d}s^2 < 0$.

From the only non-vanishing derivative $\partial g_{22}/\partial u^1 = -2\sinh(2\theta)\cosh(2\theta)$ we get as non-vanishing Christoffel symbols

$$\Gamma^1_{22} = -2\sinh(2\theta)\cosh(2\theta) \qquad \Gamma^2_{12} = \Gamma^2_{21} = \cosh(2\theta)/\sinh(2\theta). \qquad (37)$$

The element of interest of the Riemannian curvature tensor becomes $R_{1212} = -(\sinh(2\theta))^2$, and from it the scalar curvature R and the Ricci tensor R_{ij} become

$$R = -2 \qquad (R_{ij}) = -\begin{bmatrix} 1 & 0 \\ 0 & (\sinh(2\theta))^2 \end{bmatrix}. \qquad (38)$$

So H^2 with the metric equation (35) has constant negative curvature, see [7] and [9, p 5].

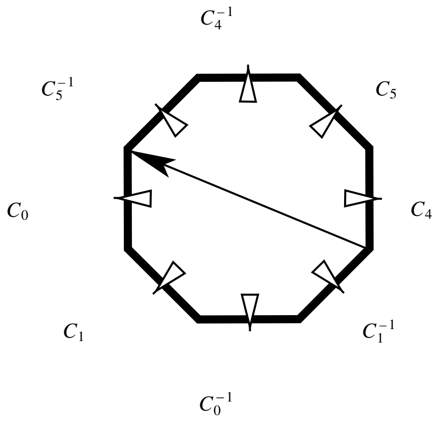

Figure 3. Schematic view of the octagon, the fundamental domain of the double torus. The images of the octagon under the four generators C_i of the homotopy group Φ_2 and their inverses C_i^{-1} are eight edge neighbours. Outward pointing arrows mark the action of these generators. The double torus can be glued from two single tori whose homotopy groups have the generators C_0, C_1 and C_4, C_5 respectively. The gluing of these two tori inside the octagon is marked by an arrowed line.

4. Topology and metric of the double torus

We wish to consider a 2D cosmological model with the topology of a double torus. The double torus may be unfolded into an octagon as described by Klein [3, pp 264–8] and by Hilbert and Cohn-Vossen [2, p 265]. This octagon in topology is denoted as the fundamental domain [7] of the double torus. The pairwise gluing of the octagon edges then reflects the topology. All the eight vertices of this octagon represent the same point of the double torus. The universal covering manifold of the double torus from this unfolding must admit a tessellation by octagons. Such a tessellation requires that eight octagons be arranged without overlap around a single vertex. This topological condition enforces the hyperbolic space H^2 and its geometry as the universal covering [7] of the octagon.

4.1. Group description of the topology

The topology of the double torus can be characterized by its homotopy group Φ_2 whose elements are closed paths starting at the same point. This homotopy group is described in [1, 2]. It has four generators with a single relation between them. Two pairs of generators are associated each to a single torus. The group relation, see equation (51) below, arises from the gluing of the two tori into the double torus along a glue line, see figure 3.

Upon embedding the octagon into its universal covering H^2, there must exist a fixpoint-free action of the homotopy group Φ_2 on H^2. This action was explicitly given by Magnus [8] and will be described in section 5.1. To any element of Φ_2 there must correspond one and only one position of an octagon on H^2. The topological fundamental domain property of the octagon now implies that any orbit on H^2 under Φ_2 has one and only one representative on a single octagon. This property assures the fundamental domain property from the point of view of group action. Points on the intersection of different octagons require an extra treatment.

We now describe the various manifolds associated to the double torus in terms of actions, subgroups and cosets of the group $SU(1, 1)$.

Select as the fundamental domain of Φ_2 the octagon X_0 centred at $x^0 = (1, 0, 0)$ on H^2, corresponding to $z_0 = 0$ on the disc, see figure 2. For any point z of this octagon there exists an element $p \in SU(1,1)/U(1) : z_0 \to z$. For an arbitrary point $c \sim z \in H^2$ interior to an octagon, the tessellation of H^2 by octagons as fundamental domains implies that there exist unique elements $\langle \phi \in \Phi_2,\ p \in SU(1,1)/U(1)\rangle$ such that

$$c = \phi\, p \qquad \phi \in \Phi_2. \tag{39}$$

Combined with equation (15) we find for arbitrary elements of $SU(1,1)$ a unique factorization

$$g = \phi\, p\, h \qquad \phi \in \Phi_2 \quad h \in U(1). \tag{40}$$

This is a double coset factorization of $SU(1,1)$ with respect to the left subgroup Φ_2 and the right subgroup $U(1)$. The group elements p which generate the points of the initial octagon X_0 are the representatives of the corresponding double cosets $\Phi_2\backslash SU(1,1)/U(1)$.

For the group actions we obtain from equation (40): under the left action of Φ_2 we find $(\tilde{\phi}, \phi\, p\, h) \to (\tilde{\phi}\,\phi)\, p\, h$ governed by group multiplication within Φ_2. The general left action of $SU(1,1)$, $(\tilde{g}, \phi\, p\, h) \to \phi'\, p'\, h'$ implies on H^2 that, starting from any fixed point of an arbitrary fixed octagon, we can find its unique image on a unique image octagon. The map $(\tilde{g}, p) \to p'$ depends on ϕ but is independent of h'. It determines a transitive action of $SU(1,1)$ on the octagon X_0 modulo the group Φ_2. The group $SU(1,1)$ with this action on X_0 is the most general one compatible with the pseudo-Riemannian metric.

4.2. Geodesics on the double torus

Consider geodesics on H^2, equipped with the pseudo-Riemannian metric of section 3.3. Viewing them as sections of the hyperboloid in $M(1,2)$ with planes through $(0,0,0)$ perpendicular to space-like vectors [9] one concludes that they are always infinite. Any such geodesic starting at a point of the initial octagon X_0 will cross a sequence of octagons. To get the geodesic on the double torus, we must pull its points back to the initial octagon. Both the geodesic property and the geodesic distance are unchanged under this pull-back. The full geodesic on the octagon will consist of all these pull-backs. Take the centre $x^0 = (1,0,0)$ of the initial octagon X_0 and a geodesic $x(\tau), x(0) = x^0$ on H^2 of fixed direction. Suppose that that geodesic on H^2 hits at $\tau = \tau'$ for the first time a centre $x^1 = x(\tau') = L(\phi)\, x(0)$ of another octagon in the tessellation. Then the pull-back of the geodesic to X_0 must hit the centre of X_0 after a finite geodesic distance and therefore must close on X_0. Thus the search for closed and finite geodesics through the centre of the initial fundamental domain is converted on the hyperbolic covering manifold H^2 into the crystallographic search for sections of geodesics, characterized by their length and direction, which connect the centres of octagons in the tessellation. In what follows we shall focus on the representative closed geodesics passing through the centre of the octagon. In sections 6.2 and 6.4 we shall determine the directions and the shortest geodesic distances between centres of octagons. In the double torus model, these geodesics become the shortest closed geodesics. As explained in the introduction and in detail in [7], the shortest closed geodesics are of interest for observing the topology of the model.

We can compute the geodesic distance from x^0 to the centre $x^1 = L(\phi)\, x^0$ of the image octagon from the group element $\phi \in \Phi_2$. When the group element is written in terms of the Euler angles equation (12) as $\phi = g(\alpha, \theta, \gamma)$, the scalar product of the vectors x_0, x_1 pointing to the two centres is determined by the second hyperbolic Euler angle as

$$\langle x^0, x^1\rangle = \langle x^0, L(\phi(\alpha,\theta,\gamma))\, x^0\rangle = \cosh(2\theta) \qquad \theta = \theta(\phi). \tag{41}$$

Any other point y^0 inside the octagon X_0 from equations (39), (40) can be reached by application of a fixed group element $p :\ y^0 = L(p)x^0$. The image y^1 of y^0 under $\phi \in \Phi_2$ is in the octagon whose centre is $L(\phi)x^0$ and located at the point $L(\phi p)x^0$. By construction, the geodesic from y^0 to its image $y^1 = \phi y^0, \phi \in \Phi_2$ is closed on the double torus. The geodesic distance from y^0 to y^1 is determined by the hyperbolic cosine

$$\langle y^0, y^1\rangle = \langle x^0, L((p^{-1}\phi p)(\alpha', \theta', \gamma'))\, x^0\rangle = \cosh(2\theta') \qquad \theta' = \theta'(p^{-1}\phi p). \tag{42}$$

Here we used the invariance of the scalar product under the Lorentz group $SO_{\uparrow}(1, 2)$. This expression generalizes equation (41). It means that the search for general closed geodesics will depend on the initial point of the octagon. In section 6.2 we shall extend the search to vertices of the octagon. An analysis of equation (42) for general points requires crystallographic elaboration and will not be given here.

The pull-back of any geodesic on H^2 which does not connect corresponding points for any pair of octagons does not close on H_0.

In the next sections we describe the details of the octagon and of the homotopy group.

5. The Coxeter and the homotopy group

We describe the homotopy group of the double torus as a subgroup of a hyperbolic Coxeter group.

5.1. The hyperbolic Coxeter group

Consider the hyperbolic Coxeter group generated by three Weyl reflections equation (19) with Weyl vectors and relations

$$\begin{aligned} &R_1 : k = (0, 0, -1),\ R_2 : k = (0, \cos(3\pi/8), \sin(3\pi/8)) \\ &R_3 : k = (\sinh(2\theta), -\cosh(2\theta), 0),\ \cosh(2\theta) = \cot(\pi/8) \\ &R_1^2 = R_2^2 = R_3^2 = e \\ &(R_1R_2)^2 = (R_2R_3)^8 = (R_3R_1)^8 = e. \end{aligned} \tag{43}$$

We follow Magnus [8, pp 81–95] who denotes the Coxeter group by $T(2, 8, 8)$. Products of Weyl reflections in $M(1, 2)$ generate Lorentz transformations

$$(R_1R_2) = L(B) \qquad (R_2R_3) = L((AB)^{-1}) \qquad (R_3R_1) = L(A) \tag{44}$$

which Magnus [8, pp 87–8] expresses as representations of elements $A, B \in SU(1, 1)$:

$$\begin{aligned} &A := \mathrm{i}/\sin(\alpha_0)\begin{bmatrix} \cos(\alpha_0) & \rho \\ -\rho & -\cos(\alpha_0)\end{bmatrix} \\ &B := \begin{bmatrix} \exp(\mathrm{i}\alpha_0) & 0 \\ 0 & \exp(-\mathrm{i}\alpha_0)\end{bmatrix} \\ &AB := \mathrm{i}/\sin(\alpha_0)\begin{bmatrix} \cos(\alpha_0)\exp(\mathrm{i}\alpha_0) & \rho\exp(-\mathrm{i}\alpha_0) \\ -\rho\exp(\mathrm{i}\alpha_0) & -\cos(\alpha_0)\exp(-\mathrm{i}\alpha_0)\end{bmatrix} \\ &\alpha_0 := \pi/8 \qquad \cot(\alpha_0) = \left(\left(1+\sqrt{1/2}\right)/\left(1-\sqrt{1/2}\right)\right)^{1/2} \\ &\rho = \sqrt{\cos(\pi/4)} = \sqrt{\sqrt{1/2}}. \end{aligned} \tag{45}$$

Here the fixed angle $\alpha_0 = \pi/8$ is taken from Magnus and should be carefully distinguished from the general Euler angle α used in equation (10). All three matrices are of elliptic type.

By use of equations (6) and (12) we find for the exponential and Euler parameters the values

$$\begin{aligned}
&A : \tilde{\alpha} = \pi/2,\ \omega(A) = (\cot(\alpha_0), -\rho/\sin(\alpha_0), 0)\\
&z(A) = \exp(\mathrm{i}\pi)\sqrt{\cos(2\alpha_0)/(1+\sin(2\alpha_0))}\\
&\Omega(A) = (\pi/2, \operatorname{arccosh}(\cot(\alpha_0)), 0)\\
&B : \tilde{\alpha} = \pi/8,\ \omega(B) = (1, 0, 0), z(B) = 0\\
&\Omega(B) = (\alpha_0, 0, 0)\\
&AB : \tilde{\alpha} = \pi - \alpha_0,\ \omega(AB) = ((\cot(\alpha_0))^2, -\rho\cos(\alpha_0)/(\sin(\alpha_0))^2, \rho/\sin(\alpha_0))\\
&z(AB) = \exp(\mathrm{i}7\pi/8)\sqrt{\cos(2\alpha_0)}\\
&\Omega(AB) = (\pi/2, \operatorname{arccosh}(\cot(\alpha_0)), \alpha_0).
\end{aligned} \tag{46}$$

The points $z(g) = (\omega_1 + \mathrm{i}\omega_2)/(1+\omega_0)$, $g = A, B, AB$ represent the fixpoints $\omega(g)$ in the hyperbolic disc model.

The matrices A, B, AB have the following properties which are relevant under the homomorphic map to the Lorentz group:

$$A^2 = -e \qquad B^8 = -e \qquad (AB)^8 = -e. \tag{47}$$

The vectors $\omega(B), \omega(AB), \omega(A)$ are the vertices of the triangular fundamental domain on H^2 for the Coxeter group $T(2, 8, 8)$, see figure 2. The hyperbolic edge lengths of this triangle are given from equations (41) and (46) and from hyperbolic trigonometry, [9, p 86], by

$$\begin{aligned}
&\langle\omega(B), \omega(AB)\rangle =: \cosh(2\theta(B, AB)) = (\cot(\alpha_0))^2\\
&\langle\omega(AB), \omega(A)\rangle =: \cosh(2\theta(AB, A)) = \cot(\alpha_0)\\
&\langle\omega(A), \omega(B)\rangle =: \cosh(2\theta(A, B)) = \cot(\alpha_0).
\end{aligned} \tag{48}$$

The hyperbolic angle $2\theta(B, AB)$ is defined between the unit vectors $\omega(B), \omega(AB)$. The hyperbolic triangle has two edges of equal length which form an angle $\pi/2$, and two more angles $\pi/8$.

5.2. The homotopy group

With Magnus [8] we pass from hyperbolic triangles to octagons and from the hyperbolic Coxeter group to the homotopy group Φ_2. The octagon, the fundamental domain of the group Φ_2, is obtained by applying all reflections in the planes containing $\omega(B) = (1, 0, 0)$ and one of the vectors $(\omega(A), \omega(AB))$. It consists of 16 triangles. The vertices of the octagon are obtained from $\omega(AB)$, the midpoints of edges from $\omega(A)$ by applying all powers of $L(B)$ to these two vectors respectively. The images of the octagon under Φ_2 yield a tessellation of H^2.

In the octagon tessellation of H^2, any vertex is surrounded by eight octagons sharing that vertex. The Coxeter group equation (43) has an involutive automorphism by the interchange of the generators R_1, R_2. As a consequence, any Coxeter triangle has two equal angles, and the triangular tessellation has two classes of vertices with congruent surroundings. These classes form the centres and the vertices respectively of the octagon tessellation, see figure 2. Seen from any centre of an octagon, the hyperbolic distance of its interior points to its centre is smaller than their distance to any other octagon centre. So the octagons form hyperbolic Voronoi cells. The tessellation seen not from the octagon centres but rather from their vertex set is a copy of the octagon tessellation. The new octagons around the vertices may be called the dual Delone cells and in shape coincide with the octagons.

The homotopy group Φ_2 as a subgroup of the Coxeter group $T(2, 8, 8)$ can be generated by (an even number of) Weyl reflections in the hyperbolic edges of this triangle. Centre and vertices of the octagon are marked by full and open circles respectively. The homotopy group

Φ_2 according to Magnus [8, pp 92–3] is a subgroup of the hyperbolic Coxeter group and has the generators

$$C_0 = AB^{-2} \qquad C_1 = BC_0B^{-1} \qquad C_4 = B^4C_0B^{-4} \qquad C_5 = B^5C_0B^{-5} \tag{49}$$

which have the Euler angle parameters

$$\begin{aligned} C_0 &: \Omega = (\pi/2, \operatorname{arccosh}(\cot(\alpha_0)), -\pi/4)\\ C_1 &: \Omega = (5\pi/8, \operatorname{arccosh}(\cot(\alpha_0)), -3\pi/8)\\ C_4 &: \Omega = (\pi, \operatorname{arccosh}(\cot(\alpha_0)), -3\pi/4)\\ C_5 &: \Omega = (9\pi/8, \operatorname{arccosh}(\cot(\alpha_0)), -7\pi/8). \end{aligned} \tag{50}$$

We call this set-up the Magnus representation of the homotopy group. Note that the group Φ_2 acts without fixpoints and so, in contrast to the Coxeter group, does not contain reflections or rotations.

The generators are subject to the relation

$$(C_0C_1^{-1}C_0^{-1}C_1)(C_4C_5^{-1}C_4^{-1}C_5) = e. \tag{51}$$

As mentioned in section 4.1, this relation in topology expresses the gluing of the double torus from two single tori. A geometric interpretation of this group relation will be given in section 6.1.

5.3. The Coxeter group acting on $SU(1, 1)$

We wish to implement the Coxeter group in terms of actions on $SU(1, 1)$. Our goal is to obtain the Magnus representation equation (45) of the homotopy group and its embedding into the Coxeter group on the level of $SU(1, 1)$ without use of the Lorentz group. Another reason for this analysis can be seen by comparison with free spin-(1/2) fields in special relativity and their discrete versions [6]: these spinors under the unimodular time-preserving Lorentz group transform according to its covering group $Sl(2, C)$. Involutions outside this group like parity or Coxeter reflections for them must be introduced by automorphisms. In the present case, spinors on H^2 are two-component fields $(\chi_1(x), \chi_2(x))$. A natural action of $g \in SU(1, 1)$ on spinors is defined by

$$\begin{bmatrix} (T_g\chi_1)(x)\\ (T_g\chi_2)(x) \end{bmatrix} := g \begin{bmatrix} \chi_1(L^{-1}(g)x)\\ \chi_2(L^{-1}(g)x) \end{bmatrix}. \tag{52}$$

The operators T_g in equation (52) obey $T_{g_1} \circ T_{g_2} = T_{g_1g_2}$.

We turn now to the action of involutions on the spinors and employ the involutive automorphisms of section 2.4. To get pre-images of the Coxeter group acting on $SU(1, 1)$ we use the Weyl vectors from equation (43) and the parameters of equation (20) and define by use of equation (24) three involutive automorphisms of $SU(1, 1)$,

$$\begin{aligned} &s_1 : k = (0, 0, -1),\ 2\alpha = -\pi/2,\ \theta = 0\\ &s_1 = g_1\bar{g_1}^{-1} \circ \psi_1\\ &g_1(\bar{g_1})^{-1} = e\\ &s_2 : k = (0, \cos(3\pi/8), \sin(3\pi/8)),\ 2\alpha = 3\pi/8,\ \theta = 0\\ &s_2 = g_2\bar{g_2}^{-1} \circ \psi_1\\ &g_2(\bar{g_2})^{-1} = r_3(7\pi/8)\\ &s_3 : k = (\sinh(2\theta), -\cosh(2\theta), 0),\ 2\alpha = \pi,\ \cosh(2\theta) = \cot(\pi/8)\\ &s_3 = g_3\bar{g_3}^{-1} \circ \psi_1\\ &g_3(\bar{g_3})^{-1} = r_3(\pi/2)b_2(2\theta)r_3(\pi/2)r_3(\pi/2) = -r_3(\pi/2)b_2(2\theta)\\ &\qquad\qquad = \begin{bmatrix} -\mathrm{i}\cosh(2\theta) & -\mathrm{i}\sinh(2\theta)\\ \mathrm{i}\sinh(2\theta) & \mathrm{i}\cosh(2\theta) \end{bmatrix}. \end{aligned} \tag{53}$$

We compute the pairwise products of these automorphisms. The automorphism ψ_1 in the products cancel in pairs due to equation (23) and we obtain operators acting only by left-multiplication. We find by comparison with the matrices equation (45) in the Magnus representation the left actions

$$\begin{aligned}(s_1 \circ s_2) &= \psi_1 \circ r_3(7\pi/8) \circ \psi_1 = -r_3(\pi/8) \circ \psi_1^2 = -r_3(\pi/8) \\ &= -B,\ (s_2 \circ s_3) = (AB)^{-1},\ (s_3 \circ s_1) = -A.\end{aligned} \qquad (54)$$

The operator relations of equation (54) lift into the relations equation (44) of the Coxeter group due to $L(g) = L(-g)$.

Proposition 3. The involutive automorphisms s_1, s_2, s_3 of equation (53) with $L(s_i) := R_i,\ i = 1, 2, 3$ generate for the Coxeter group equation (43) a pre-image which acts on $SU(1,1)$. We shall speak about the Coxeter group generated by $\langle s_1, s_2, s_3\rangle$.

5.4. Conjugation of Φ_2 by the Coxeter group

The matrices A, B under conjugation with the three automorphisms s_1, s_2, s_3 of equation (53) transform as

$$\begin{array}{cccc} X & s_1Xs_1 & s_2Xs_2 & s_3Xs_3 \\ A & -A & -B^{-1}AB & -A \\ B & B^{-1} & B^{-1} & -AB^{-1}A. \end{array} \qquad (55)$$

We shall show that the homotopy group Φ_2, augmented by the element B^4, is a normal subgroup of the Coxeter group. The element $B^4, B^{-4} = -B^4$ has a simple geometric interpretation: in $M(1,2)$ it is a rotation by $2\tilde{\alpha} = \pi$ and transforms the two tori of the double torus into one another. Writing the generators equation (49) of Φ_2 in terms of the matrices A, B one finds the conjugation relations

$$B^4C_0B^{-4} = C_4 \qquad B^4C_1B^{-4} = C_5 \qquad B^4C_4B^{-4} = C_0 \qquad B^4C_5B^{-4} = C_1. \qquad (56)$$

The subgroup generated by B^4 consists of the elements $\langle e, -e, B^4, -B^4\rangle$. Only the identity element is shared with Φ_2. We denote the group generated by $\langle C_0, C_1, C_4, C_5, B^4\rangle$ as $\tilde{\Phi}_2$. It is a semidirect product with Φ_2 as the normal subgroup. Now we study the conjugation of the generators of this group under the three automorphisms equation (53) and obtain by use of equations (49), (55) and (56):

$$\begin{array}{cccc} \mathrm{i} & s_1C_is_1 & s_2C_is_2 & s_3C_is_3 \\ 0 & -C_0B^4 & C_5^{-1} & -C_0^{-1} \\ 1 & C_5^{-1} & C_4^{-1} & -C_0C_1C_4^{-1}B^4 \\ 4 & -C_4B^4 & C_1^{-1} & C_0C_4^{-1}C_0^{-1} \\ 5 & C_1^{-1} & C_0^{-1} & C_0C_5C_4^{-1}B^4. \end{array} \qquad (57)$$

We also find from equation (55)

$$s_1B^4s_1 = -B^4 \qquad s_2B^4s_2 = -B^4 \qquad s_3B^4s_3 = -C_0C_4^{-1}B^4. \qquad (58)$$

All these conjugations yield again elements of the group $\tilde{\Phi}_2$. The inclusion of B^4 is necessary since it appears in the conjugations of equation (57). Therefore, we find the following.

Proposition 4. The extension $\tilde{\Phi}_2$ of the homotopy group Φ_2 is a normal subgroup of the Coxeter group generated by $\langle s_1, s_2, s_3\rangle$.

Corresponding relations hold true on the level of the Lorentz group. In view of this property under conjugation we can call the hyperbolic Coxeter group the discrete symmetry group of the octagonal tessellation associated with the double torus.

6. The homotopy group and closed geodesics

With the information on the homotopy group, its action and symmetry obtained in section 5 we return to the octagonal tessellation and analyse the closed geodesics as outlined in section 4.

6.1. The action of the generators of Φ_2

By application of the four generators of Φ_2 and their inverses to an initial octagon X_0 whose centre is located at $z = 0$, we obtain eight images. It turns out that all of them are different edge neighbours of X_0. For $C_0 = AB^{-2}$ we have $\theta(C_0) = \theta(A)$ and from equation (46) $\cosh(\theta(A)) = \cot(\alpha_0)$. From these expressions we obtain for the geodesic distance equation (41)

$$\langle x^0, L(C_0)x^0\rangle = \cosh(2\theta(C_0)) = 2(\cot(\alpha_0))^2 - 1. \tag{59}$$

Since $L(A)$ is a rotation by π in the midpoint of an edge of the central octagon, the image under C_0 is an edge neighbour. The other generators equation (49) of Φ_2 are conjugates of C_0 by powers of B. Therefore, all the corresponding images of the central octagon under the generators are edge neighbours. All pairs X_0, $L(\phi)X_0$ of octagons are shown in figure 3. In figures 3 and 4 we omit the hyperbolic geometry of the octagons as displayed in figure 2.

The action of the generators of Φ_2 described so far refers to the full hyperbolic disc as the covering manifold of the double torus. The relation to the homotopy group as described by closed paths on the double torus arises as follows: take a homotopic path P which passes from X_0 into $L(\phi)X_0$ through a shared edge. Mark first this shared entrance edge as seen from the centre of $L(\phi)X_0$. Then identify the pre-image of this entrance edge seen from the centre of X_0. The reentry path P into X_0 passes through this edge. This information is presented in figure 3. The path in the homotopy group for any generator may in fact be represented on the covering manifold by a geodesic section between two octagons that share an edge, see figure 4.

We would like to use the information on the action of single generators to find products of generators which, when applied to the central octagon, give a sequence of pairwise edge-sharing images of X_0 around a vertex.

The action of a generator $g_i = C_j$ is always obtained from products of Weyl reflections in the edges of the fixed Coxeter triangle in the central octagon shown in figure 2. We call it a passive transformation. Consider a word $g = g_1g_2g_3\ldots$ and rewrite it by use of conjugations as

$$\begin{aligned} g = g_1g_2g_3\ldots &= \ldots((g_1g_2)g_3(g_1g_2)^{-1})(g_1g_2g_1^{-1})g_1 \\ &= \ldots g_3^*g_2^*g_1^*. \end{aligned} \tag{60}$$

The word on the left-hand side is expressed on the right-hand side as a product of conjugates g_j^* whose terms appear in reverse order. When acting on the initial central octagon say X_0, the term g_i^* conjugate to g_i on the right-hand side acts on the image $L(g_1g_2\ldots g_{i-1})X_0$ of X_0. Therefore, the right-hand side of equation (60) expresses a sequence of active transformations, and we learn that these active transformations appear in the reverse order of the sequence of passive transformations.

We apply the active transformations to the passage around the selected vertex $z = \tanh(\theta(AB))\exp(\mathrm{i}7\pi/8)$ of X_0. We use the geometric information shown in figures 3 and 4 to pass counterclockwise around this vertex through a sequence of edges. We find the sequences

$$\begin{aligned} &C_5^{-1}, C_4C_5^{-1}, C_5C_4C_5^{-1}, C_4^{-1}C_5C_4C_5^{-1}, C_1^{-1}C_4^{-1}C_5C_4C_5^{-1}, C_0C_1^{-1}C_4^{-1}C_5C_4C_5^{-1}, \\ &\qquad C_1C_0C_1^{-1}C_4^{-1}C_5C_4C_5^{-1}, C_0^{-1}C_1C_0C_1^{-1}C_4^{-1}C_5C_4C_5^{-1}. \end{aligned} \tag{61}$$

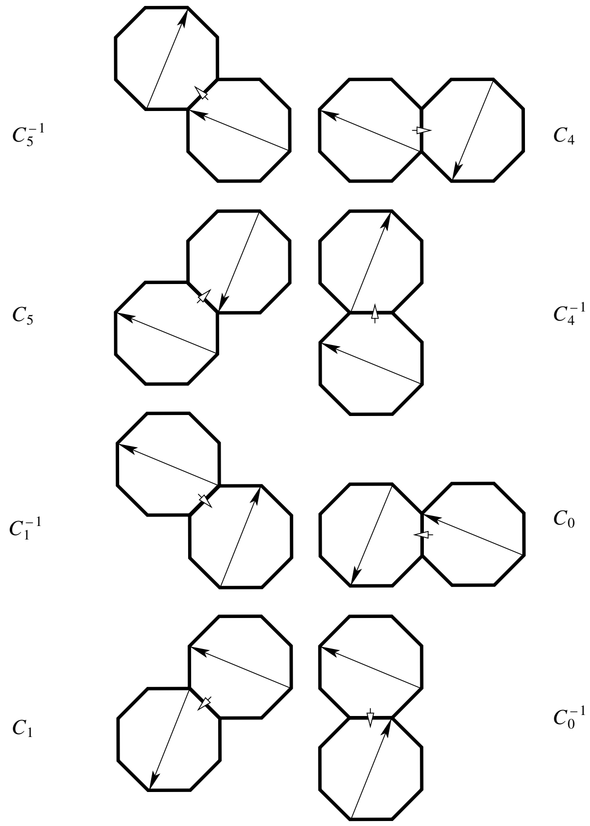

Figure 4. Schematic view of the action of the generators of the homotopy group Φ_2 on the central octagon. The arrowed glue line, see figure 3, is marked inside each octagon. The image of the central octagon under any generator or its inverse is an edge neighbour. The direction of the map is indicated by a white arrow passing through the edge. A homotopic path on the covering manifold could be taken as a geodesic connecting the centres of octagons sharing an edge. The sequence of generators as shown, multiplied from left to right, produces a sequence of images which share a vertex.

which should finally restore the octagon X_0. The full passive sequence from equation (61) is

$$g = C_5^{-1}C_4C_5C_4^{-1}C_1^{-1}C_0C_1C_0^{-1} = e. \tag{62}$$

It becomes the identity by using the single relation equation (51) of the group Φ_2. We have found a geometric interpretation of this relation on H^2: the subwords of this relation in the Magnus representation describe active transformations of the octagon around one of its vertices. Similar results apply to all eight vertices, and we list them in table 1. Moreover we compute the geodesic distance from the central octagon to these neighbours in table 2.

6.2. Vertex neighbours

We shall use the self-dual property of the octagon tessellation explained in section 5.2 as follows: instead of analysing eight octagon centres around a single vertex we can equally well consider eight vertices around a single centre. Instead of geodesics passing through octagon centres we now choose those passing through octagon vertices.

Table 1. The words w_ν which transform the octagon into images sharing a vertex $z_\nu = \sqrt{\cos(2\alpha_0)}\exp(\mathrm{i}(2\nu-1)\pi/8)$. The full words yield the identity element due to the group relation equation (51). The seven images of the octagon around the chosen vertex appear in counterclockwise order if each word is cut after 1, 2, ..., 7 entries, counted from left to right.

ν	w_ν
0	$C_1^{-1}C_0C_1C_0^{-1}C_5^{-1}C_4C_5C_4^{-1}$
1	$C_4C_5C_4^{-1}C_1^{-1}C_0C_1C_0^{-1}C_5^{-1}$
2	$C_5C_4^{-1}C_1^{-1}C_0C_1C_0^{-1}C_5^{-1}C_4$
3	$C_4^{-1}C_1^{-1}C_0C_1C_0^{-1}C_5^{-1}C_4C_5$
4	$C_5^{-1}C_4C_5C_4^{-1}C_1^{-1}C_0C_1C_0^{-1}$
5	$C_0C_1C_0^{-1}C_5^{-1}C_4C_5C_4^{-1}C_1^{-1}$
6	$C_1C_0^{-1}C_5^{-1}C_4C_5C_4^{-1}C_1^{-1}C_0$
7	$C_0^{-1}C_5^{-1}C_4C_5C_4^{-1}C_1^{-1}C_0C_1$

Table 2. Hyperbolic cosine for geodesic distances of vertices of a single octagon.

ν	$\nu\pi/4$	$\cos(\nu\pi/4)$	$\cosh(2\theta(P_0, P_\nu))$	Multiplicity
0	0	1	1	
1, 7	$\pm\pi/4$	$\sqrt{1/2}$	$(1-\sqrt{1/2})^{-2}(-1/2+2\sqrt{1/2})$	8
2, 6	$\pm\pi/2$	0	$(1-\sqrt{1/2})^{-2}(3/2+2\sqrt{1/2})$	16
3, 5	$\pm 3\pi/4$	$-\sqrt{1/2}$	$(1-\sqrt{1/2})^{-2}(7/2+2\sqrt{1/2})$	16
4	$\pm\pi$	-1	$(1-\sqrt{1/2})^{-2}(3/2+6\sqrt{1/2})$	8

Choosing a geodesic starting at a fixed vertex, we may require it to run inside the octagon and hit another vertex. In this way we can avoid any pull-back for these shortest geodesics. These geodesics are dual to the geodesics between the centres of octagons sharing a vertex.

The positions of the eight vertices seen from the centre $z = 0$, see figure 2, are given from equations (45) and (46) by

$$\begin{aligned} z_\nu &= \tanh(\theta(AB))\exp(\mathrm{i}\pi(2\nu-1)/8) \\ &= \sqrt{\cos(2\alpha_0)}\exp(\mathrm{i}\pi(2\nu-1)/8) \qquad \nu = 1, \ldots, 7. \end{aligned} \tag{63}$$

From equation (33) we obtain by the insertions $z_1 \to z_0$, $z_2 \to z_\nu$ for the centre of the geodesic circle that connects z_0 with z_ν, $\nu \neq 0$:

$$q(z_0, z_\nu) = \exp(\mathrm{i}\pi(\nu-1)/8)\ \mathrm{cotanh}(2\theta(AB))/\cos(\pi\nu/8). \tag{64}$$

We compute the hyperbolic cosine for all pairs of vertices and find from hyperbolic trigonometry, see [9, pp 83–91],

$$\begin{aligned} \cosh(2\theta(P_0, P_\nu)) &= \cosh(2\theta(P_0))\cosh(2\theta(P_\nu)) - \sinh(2\theta(P_0))\sinh(2\theta(P_\nu))\cos(\nu\pi/4) \\ &= (\cot(\alpha))^4(1-\cos(\nu\pi/4)) + \cos(\nu\pi/4). \end{aligned} \tag{65}$$

The results are given in table 2 and in figure 5. The multiplicity counts geodesic sections of equal length and starting point but of different directions which result from the octagonal symmetry. The cases $\nu = 1, 7$ yield the same distance as between centres of edge neighbours. The point P_4 results from a rotation by π and yields the largest hyperbolic distance corresponding to a vertex neighbour.

We claim that the four octagons closest to the central octagon are four vertex neighbours. Therefore, they determine the shortest closed geodesics of the double torus.

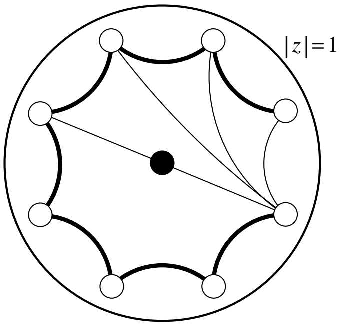

Figure 5. Examples of four shortest closed geodesics in the octagon model as part of geodesic circles between vertices (open circles) of the central octagon. They start at the vertex $\nu = 0$ and end at the vertices $\nu = 1, 2, 3, 4$ respectively. For $\nu = 1$ the geodesic is an edge of the octagon. It corresponds dually to a geodesic between centres of edge neighbours.

Table 3. Products of pairs of generators of Φ_2 up to inversion.

i, j	C_iC_j	$C_i(C_j)^{-1}$	$(C_i)^{-1}C_j$
0, 1	$AB^{-1}AB^{-3}$	$-ABAB^{-1}$	$-B^2ABAB^{-3}$
1, 0	$BAB^{-3}AB^{-2}$	$-BAB^{-1}A$	$-B^3AB^{-1}AB^{-2}$
0, 4	AB^2AB^{-6}	$-AB^4AB^{-4}$	$-B^2AB^4AB^{-6}$
4, 0	$B^4AB^{-6}AB^{-2}$	$-B^4AB^{-4}A$	$-B^6AB^{-4}AB^{-2}$
0, 5	AB^3AB^{-7}	$-AB^5AB^{-5}$	$-B^2AB^5AB^{-7}$
5, 0	$B^5AB^{-7}AB^{-2}$	$-B^5AB^{-5}A$	$-B^7AB^{-5}AB^{-2}$
1, 4	$BABAB^{-6}$	$-BAB^3AB^{-4}$	$-B^3AB^3AB^{-6}$
4, 1	$B^4AB^{-5}AB^{-3}$	$-B^4AB^{-3}AB^{-1}$	$-B^6AB^{-3}AB^{-3}$
1, 5	BAB^2AB^{-7}	$-BAB^4AB^{-5}$	$-B^3AB^4AB^{-7}$
5, 1	$B^5AB^{-6}AB^{-3}$	$-B^5AB^{-4}AB^{-1}$	$-B^7AB^{-4}AB^{-3}$
4, 5	$B^4AB^{-1}AB^{-7}$	$-B^4ABAB^{-5}$	$-B^6ABAB^{-7}$
5, 4	$B^5AB^{-3}AB^{-6}$	$-B^5AB^{-1}AB^{-4}$	$-B^7AB^{-1}AB^{-6}$
0, 0	$AB^{-2}AB^{-2}$	e	e
1, 1	$BAB^{-2}AB^{-3}$	e	e
4, 4	$B^4AB^{-2}AB^{-6}$	e	e
5, 5	$B^5AB^{-2}AB^{-7}$	e	e

6.3. Products of generators

We consider products of the generators C_i, equation (49). All possible products of two of them up to some inversions can be written in terms of the matrices A, B, equation (45), as given in table 3.

6.4. Geodesic distances for products of generators

We wish to have a geometric interpretation for the action of products of generators on the octagon. Consider the product C_iC_j of two generators. If we define $C_jX_0 := X_j$ as a reference octagon we have $X_0 = C_j^{-1}X_j$, $(C_iC_j)X_0 = C_iX_j$. Therefore, we find that X_0 and $(C_iC_j)X_0$ are edge neighbours to the single octagon X_j. When running through all products of generators we run through all pairs of different edge neighbours to a single octagon. We expect to find only four different geodesic distances between such pairs.

We now compute for the products of generators the geodesic distances from the matrix products of table 3. It suffices to compute the second hyperbolic Euler angle for the matrices

Table 4. Hyperbolic cosine for geodesic distances of octagon images under the products of table 3.

μ	$\mu\pi/4+\pi$	$\cos(\mu\pi/4+\pi)$	$\cosh(2\theta(AB^{\mu}A))$
0	π	-1	1
1, 7	$5\pi/4, 3\pi/4$	$-\sqrt{1/2}$	$(1-\sqrt{1/2})^{-2}(3/2+2\sqrt{1/2})$
2, 6	$3\pi/2, 5\pi/2$	0	$(1-\sqrt{1/2})^{-2}(11/2+6\sqrt{1/2})$
3, 5	$7\pi/4, 9\pi/4$	$\sqrt{1/2}$	$(1-\sqrt{1/2})^{-2}(19/2+10\sqrt{1/2})$
4	2π	1	$\cosh(4\theta(A)) = (1-\sqrt{1/2})^{-2}(19/2+14\sqrt{1/2})$

$AB^{\mu}A$, $\mu = 1, \ldots, 7$ since all the group elements in table 2 arise from $AB^{\mu}A$ by right- and left-multiplication with powers of B. To these triple products we apply equations (13) and (14) and use the Euler angle parameters from equation (46). First we find from $AB^{\mu}A$ for the angle corresponding to $2\gamma_1$ in equation (13): $\cos(2\gamma_1) = \cos(\mu\pi/4+\pi)$ with values given in column 3 in table 4. From equations (14) and (46) we then get

$$\begin{aligned}\cosh(2\theta(AB^{\mu}A)) &= (\cosh(2\theta(A)))^2 + (\sinh(2\theta(A))^2\cos(\mu\pi/4+\pi) \\ &= (2(\cot(\alpha))^2-1)^2(1+\cos(\mu\pi/4+\pi)) - \cos(\mu\pi/4+\pi) \qquad (66)\end{aligned}$$

which gives table 4.

From the expressions in table 1 we can infer in particular all the products of two generators which move an octagon counterclockwise around a vertex. They are: $C_1^{-1}C_0, C_0C_1, C_1C_0^{-1}, C_0^{-1}C_5^{-1}, C_5^{-1}C_4, C_4C_5, C_5C_4^{-1}, C_4^{-1}C_1^{-1}$. All of them from table 3 correspond to $\mu = 7$. We therefore expect that these cases in table 4 yield the same geodesic distance as $\nu = 2$ in table 2 which is the case. The case $\mu = 4$ in table 4 gives twice the geodesic distance of an edge neighbour, the centres of the three octagons are on a single geodesic. The pull-back of this geodesic section runs twice over the shortest closed geodesic.

7. Conclusion and outlook

The double torus provides a model for a 2D octagonal cosmos with a topology of genus 2. We study the geometry of this model by passing to its universal covering, the hyperbolic space H^2. Insight into the model is gained from the action of groups and subgroups. The continuous group $SU(1,1)$ acts on H^2 and, modulo the homotopy group Φ_2, on the octagon. This action determines the homogeneity of both spaces. A discrete hyperbolic Coxeter group acts as a subgroup on $SU(1,1)$. An extension of the homotopy group Φ_2 of the double torus is a normal subgroup of this hyperbolic Coxeter group. So the hyperbolic Coxeter group expresses by conjugation the discrete symmetry of the octagonal tessellation of H^2. Representative closed geodesics of the double torus model have on H^2 the crystallographic interpretation of geodesic links between octagon centres or vertices. Four shortest geodesic sections connect pairs of vertices of the octagon. The directions and geodesic distances for these and the next closed geodesics are explicitly computed.

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References

[1] Coxeter H S M and Moser W O 1965 *Generators and Relations for Discrete Groups* (New York: Springer)
[2] Hilbert D and Cohn-Vossen S 1932 *Anschauliche Geometrie* (Berlin: Springer)
[3] Klein F 1928 *Vorlesungen über nicht-euklidische Geometrie* (Berlin: Springer)
[4] Kramer P 1993 Algebraic structures for one-dimensional quasiperiodic systems *J. Phys. A: Math. Gen.* **26** 213–28
[5] Lorente M and Kramer P 1995 Non-Euclidean crystallography *Symmetries in Science* vol 7, ed B Gruber (New York: Plenum) pp 315–29
[6] Lorente M and Kramer P 1999 Representations of the discrete inhomogeneous Lorentz group and Dirac wave equation on the lattice *J. Phys. A: Math. Gen.* **32** 2481–97
[7] Lachieze-Rey M and Luminet J-P 1995 Cosmic topology *Phys. Rep.* **254** 135–214
[8] Magnus J W 1974 *Non-Euclidean Tesselations and Their Groups* (New York: Academic)
[9] Ratcliffe J G 1994 *Foundations of Hyperbolic Manifolds* (New York: Springer)
[10] Seifert H and Threlfall W 1980 *Lehrbuch der Topologie* (New York: Chelsea)
[11] Wolf J A 1984 *Spaces of Constant Curvature* (Wilmington: Publish or Perish)